Boundary Feedback Control of Unstable Heat Equation with Space and Time Dependent Coefficient

Junji Jia

Department of Applied Mathematics

Room 210, Middlesex College, The University of Western Ontario

London, Ontario, N6A 5B7, Canada

Email: jjia5@uwo.ca

Abstract

In this paper, we study the control of the linear heat equation with a space and time dependent coefficient function by the Dirichlet and Neumann boundary control laws. This equation models the heat diffusion and space, time dependent heat generation in a one dimensional rod. Without control, the system is unstable if the coefficient function is positive and large. With boundary control based on the state feedback, we show that for the time analytic coefficient $a(x,t)$, the exponential stability of the system at any rate can be achieved. It is further shown that both the control of the Dirichlet and Neumann boundary value systems can be stabilized using this method. In doing this, the control kernels are *explicitly* calculated as series of approximation and they are used in the simulations. The numerical simulation confirmed the theoretical arguments and the controllability of the system. The possible future works are discussed at the end. This is the continuation of the recent work of Liu [*SIAM J. Cont. Optim.* vol. 42, pp. 1033-1043] and Smyshlyaev and Krstic [accepted by *Automatica*].

Keywords: heat equation, boundary control, stabilization.

AMS Subject Classification. 35K05, 93D15.

1 Introduction

In this paper, we continue the study of feedback control of an unstable heat equation

$$u_t(x,t) = u_{xx}(x,t) + au(x,t) \text{ in } (0,1). \tag{1.1}$$

This equation arises from the heat generation and diffusion of a rod in which the heat generation is only only sensitive to the position x but also the time t. A simple version of this feedback control problem where $a = \mu$, a constant, was first addressed by Boskovic, Krstic and Liu in [1] and the instability when μ is large was shown there. Later, Balogh and Krstic [2, 3] studied the case where $a = a(x)$, a function dependent on x. Since they used a backstepping approximation method, the same problem was reconsidered and solved by Liu [4] using a transformation skill. More recently, the generalization to $a = a(t)$ was explicitly considered by Smyshlyaev and Krstic [5]. In our paper, we generalize it to the case of $a = a(x,t)$.

For the theories that use boundary feedback control to stabilize the parabolic equations, one can refer to [1] and [3] and references there. In comparison of the existing literature, the novelty of this paper is the "explicit" construction of the feedback laws of $a = a(x,t)$ case for some $a(x,t)$ and the proof that both the kernel for the Dirichlet and Neumann boundary control problem exist.

The paper is organized as follows. Section 2 is devoted to the stabilization of Dirichlet boundary value problems of $a = a(x,t)$ case, Section 3 to the stabilization of Neumann boundary value problems. In Section 4, the simulation of $a(x,t) = x(bt + c)$ is studied and the validity of the control are shown. In section 5, we give some remarks of interesting phenomena, which may be helpful for the future works.

In what follows, we denote the region $(0,1)_x$ by X, $(0,1)_t$ by T, $0 \le y \le x \le 1$ by Ω and $X \times T$ by XT and $\Omega \times T$ by ΩT and so on. We also denote by $H^s \!:\! XT$ the usual Sobolev space (see, e.g., [6, 10]) for any $s \in \mathbf{R}$. For $s \ge 0$, $H_0^s \!:\! XT$ denotes the completion of $C_0^\infty \!:\! XT$ in $H^s \!:\! XT$, where $C_0^\infty \!:\! XT$ denotes the space of all infinitely differentiable functions on XT with compact support in XT. We denote by $\| \cdot \|$ the norm of $L^2 \!:\! XT$ with respect to space variable x. $C^n \!:\! XT$ denotes the space of all n times continuously differentiable functions on XT.

2 Dirichlet Boundary Conditions

It is well known that the Dirichlet boundary value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + a(x,t)u(x,t) & \text{in } XT, \\ u(0,t) = u(1,t) = 0 & \text{in } T \end{cases} \tag{2.1}$$

is unstable if function a is positive and large [4].

To design a boundary feedback law to stabilize the above system for function $a(x, t)$, we consider the problem

$$\begin{cases} k_{xx}(x, y, t) - k_{yy}(x, y, t) - k_t(x, y, t) = (a(y, t) + \lambda)k(x, y, t), & \text{in } \Omega T \\ k(x, 0, t) = 0, & \text{in } XT \\ k_x(x, x, t) + k_y(x, x, t) + \frac{d}{dx}(k(x, x, t)) = a(x, t) + \lambda, & \text{in } XT \end{cases} \quad (2.2)$$

where λ is any positive constant. Why we consider this problem will becomes clear in Lemma 2.2. For the moment, let us assume this problem has a unique solution k for some a. Using the solution k, we then obtain a Dirichlet boundary feedback law

$$u(1, t) = -\int_0^1 k(1, y, t)u(y, t)dy \qquad \text{in } T \qquad (2.3)$$

and Neumann boundary feedback law

$$u_x(1, t) = -k(1, 1)u(1, t) - \int_0^1 k_x(1, y, t)u(y, t)dy \qquad \text{in } T. \qquad (2.4)$$

With one of the boundary feedback laws, the system

$$\begin{cases} u_t(x, t) = u_{xx}(x, t) + a(x, t)u(x, t) & \text{in } XT, \\ u(0, t) = 0 & \text{in } T, \\ u(x, 0) = u^0(x) & \text{in } T \end{cases} \quad (2.5)$$

is exponentially stable. In this controlled system, the left hand end of a rod is insulated while the temperature or the heat flux at the other end is adjusted according to the measurement of k-weighted averaged temperature over the whole rod. Physically, if the destabilizing heat is generating inside the rod, then we cool the right end of the rod so that it is not overheated. To state this result, we also introduce the compatible conditions for the initial data

$$u^0(0) = 0, \quad u^0(1, t) = -\int_0^1 k(1, y, t)u^0(y)dy \qquad (2.6)$$

$$u^0(0) = 0, \quad u_x^0(1, t) = -k(1, 1)u^0(1, t) - \int_0^1 k_x(1, y, t)u^0(y)dy \qquad (2.7)$$

Theorem 2.1. *Assume that $\lambda > 0$ is any positive constant and $a(x, t)$ is continuous in x and analytic in t uniformly in x. For arbitrary initial data $u^0(x) \in H^1 X$ with compatible condition (2.6) or (2.7), equation (2.5) with either (2.3) or (2.4) has a unique solution that satisfies*

$$\|u(t)\|_{H^1} \le M\|u^0\|_{H^1}e^{-\lambda t}, \quad t \in (0, t_0), t_0 < 1, \qquad (2.8)$$

where M is a positive constant independent of u^0.

The idea of proving the theorem is to carefully construct a transformation

$$w(x, t) = u(x, t) + \int_0^x k(x, y, t)u(y, t)dy$$

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to convert the system (2.5) with either (2.3) or (2.4) into the exponentially stable system

$$\begin{cases} w_t = w_{xx} - \lambda w & \text{in } XT, \\ w(0,t) = w(1,t) = 0 & \text{in } T, \\ w(x,0) = w^0(x) & \text{in } X, \end{cases} \tag{2.9}$$

or

$$\begin{cases} w_t = w_{xx} - \lambda w & \text{in } XT, \\ w(0,t) = w_x(1,t) = 0 & \text{in } T, \\ w(x,0) = w^0(x) & \text{in } X, \end{cases} \tag{2.10}$$

where $w^0(x) = u^0(x) + \int_0^x k(x,y,t)u^0(y)dy$. This will be done by the following lemmas. The first one of these lemmas is due to Colton [7, 8]. We conveniently rewrite its statement and quote the proof here for reference.

Lemma 2.1. (Colton) *Suppose that $a(x,t)$ is continuous in x and analytic in t uniformly in x. Then problem (2.2) has a unique solution which is twice continuously differentiable in x, y.*

Proof. Using the substitutions

$$x = \xi + \eta, \ y = \xi - \eta, \ a(y,t) = a(\xi - \eta, t), \ G(\xi, \eta, t) = k(x,y,t),$$

the problem (2.2) is transformed to

$$\begin{cases} G_{\xi\eta}(\xi,\eta,t) = \left(a(\xi - \eta, t) + \lambda + \frac{\partial}{\partial t}\right)G(\xi,\eta,t) & 0 \le \eta \le \xi \le 1, t \in T \\ G(\xi,\xi,t) = 0 & 0 \le \xi \le 1, t \in T \\ \frac{\partial}{\partial \xi}(G(\xi,0,t) = \frac{1}{2}(a(\xi,t) + \lambda) & 0 \le \xi \le 1, t \in T. \end{cases}$$

Integrating over ξ and η, we can find

$$G(\xi,\eta,t) = \frac{1}{2}\int_\eta^\xi (a(\tau,t) + \lambda)\, d\tau + \int_\eta^\xi \int_0^\eta \left(a(\tau - s, t) + \lambda + \frac{\partial}{\partial t}\right)G(\tau,s,t)dsd\tau . \tag{2.11}$$

Then Colton showed that the above equation has a solution using the "method of dominants". This method works as follows. If we are given two series

$$S_1 = \sum_{n=1}^\infty a_{1n}t^n, \ S_2 = \sum_{n=1}^\infty a_{2n}t^n, \ t \in (0,1), \tag{2.12}$$

where $a_{2n} \ge 0$, then we say S_2 *dominates* S_1 if $|a_{1n}| \le a_{2n}, \ n = 1, 2, 3, \cdots$, and write $S_1 \ll S_2$. It can be easily checked that

$$\text{if } S_1 \ll S_2, \quad \text{then } |S_1| \le S_2, \tag{2.13}$$

$$\text{and } \frac{\partial S_1}{\partial t} \ll \frac{\partial S_2}{\partial t}, \ S_1 \ll S_2(1-t)^{-1}; \tag{2.14}$$

$$\text{if } S_1 \ll S_2, S_2 \ll S_3, \quad \text{then } S_1 \ll S_3; \tag{2.15}$$

$$\text{if } S_1 \ll S_2, S_3 \ll S_4, \quad \text{then } S_1 + S_2 \ll S_3 + S_4. \tag{2.16}$$

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Moreover, the property of "dominant" can also be kept if the integrals of the two series are not with respect to t but other variables, that is

$$\text{if } S_1 \ll S_2, \text{ then } \int_a^b S_1 dx \ll \int_a^b S_2 dx. \tag{2.17}$$

On the other hand, if a function f is analytic in $t \in (0,1)$, then there exist a positive constant C such that $f \ll C(1-t)^{-1}$.

Using this method in our problem, it can be shown that equation (2.11) has a unique twice continuously differentiable solution if a is analytic in t. In fact, since $a(x,t)$ is analytic, we can let C be a positive constant such that we have

$$a(\xi - \eta, t) + \lambda \ll C(1-t)^{-1} .$$

By the properties of dominant (2.13)-(2.17), it can be shown by induction that for the following series

$$G(\xi, \eta, t) = \sum_{n=0}^{\infty} G_n(\xi, \eta, t) \tag{2.18}$$

where

$$G_0(\xi, \eta, t) = \frac{1}{2} \int_\eta^\xi (a(\tau, t) + \lambda) \, d\tau$$

$$G_{n+1}(\xi, \eta, t) = \int_\eta^\xi \int_0^\eta \left(a(\tau - s, t) + \lambda + \frac{\partial}{\partial t} \right) G_n(\tau, s, t) ds d\tau ,$$

we have

$$G_n \ll \frac{2^n C^n \xi^n \eta^n}{n!} (1-t)^{-(n+1)}$$

and hence

$$|G_n| \leq \frac{2^n C^n \xi^n \eta^n}{n!} (1-t)^{-(n+1)} . \tag{2.19}$$

Thus the series (2.18) converges absolutely and uniformly. Moreover, since a is C^1, G is clearly C^2 in x, y.

$\square$

Remark 2.1. The proof of Lemma 2.1 provides a numeric computation scheme of successive approximation to compute the kernel function k in our feedback laws (2.3) and (2.4). This makes the feedback laws (2.3) and (2.4) very useful in real problems.

Lemma 2.2. *Let $k(x, y, t)$ be the solution of problem (2.2) and define the linear bounded operator $K : H^i \!:\! XT_0 \to H^i \!:\! XT_0$ $(i = 0, 1, 2)$ where T_0 denotes $(0, t_0)(t_0 < 1)$ by*

$$w(x,t) = (Ku)(x,t) = u(x,t) + \int_0^x k(x, y, t) u(y, t) dy, \quad \text{for } u \in H^i \!:\! XT_0. \tag{2.20}$$

Then,

1. *K has a linear bounded inverse $K^{-1}: \ H^i{:}XT_0 \to H^i{:}XT_0 \ (i = 0, 1, 2)$.*

2. *K converts the system (2.3) and (2.5) and system (2.4) and (2.5) into (2.9) and (2.10), respectively.*

Proof. To prove *1.*: that (2.20) has a bounded inverse, we set

$$v(x, t) = \int_0^x k(x, y, t) u(y, t) dy$$

and then

$$w(x, t) = u(x, t) + v(x, t).$$

Hence we have

$$\begin{aligned} v(x, t) &= \int_0^x k(x, y, t)[w(y, t) - v(y, t)] dy \\ &= \int_0^x k(x, y, t) w(y, t) dy - \int_0^x k(x, y, t) v(y, t) dy. \end{aligned} \qquad (2.21)$$

To show that this equation has a unique continuous solution, we set

$$\begin{aligned} v_0(x, t) &= \int_0^x k(x, y, t) w(y, t) dy, \\ v_n(x, t) &= - \int_0^x k(x, y, t) v_{n-1}(y, t) dy \end{aligned}$$

Though from (2.19) we see that $k(x, y, t)$ is divergent with t approaches 1^-, it is still clear that under the requirement $t \in (0, t_0)$, the $k(x, y, t)$ is still bounded in ΩT_0. Thus we can denote $M = \sup\limits_{0 \leq y \leq x \leq 1, t \in T_0} |k(x, y, t)|$. Then one can see

$$\begin{aligned} |v_0(x, t)| &\leq M \|w\|, \\ |v_1(x, t)| &\leq M^2 \|w\| x, \\ |v_2(x, t)| &\leq \frac{M^3 \|w\|}{2!} x^2, \end{aligned}$$

and by induction,

$$|v_n(x, t)| \leq \frac{M^{n+1} \|w\|}{n!} x^n.$$

These estimates show that the series

$$v(x, t) = \sum_{n=0}^{\infty} v_n(x, t)$$

converges absolutely and uniformly in XT_0 and that its sum is a continuous solution of equation (2.21). Moreover, there exists a constant $C > 0$ such that

$$\|v\| \leq C \|w\|. \qquad (2.22)$$

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This implies that there exists a bounded linear operator $\Phi : L^2 : XT_0 \to L^2 : XT_0$ such that

$$v(x,t) = (\Phi w)(x,t)$$

and then

$$u(x,t) = w(x,t) - v(x,t) = ((I - \Phi)w)(x,t) = (K^{-1}w)(x,t). \tag{2.23}$$

It is clear that $K^{-1} : L^2 : XT_0 \to L^2 : XT_0$ is bounded. To show that $K^{-1} : H^1 : XT_0 \to H^1 : XT_0$ is bounded, we take derivative in (2.21) and obtain

$$
\begin{aligned}
&v_x(x,t) \\
&= k(x,x,t)w(x,t) + \int_0^x k_x(x,y,t)w(y,t)dy - k(x,x,t)v(x,t) - \int_0^x k_x(x,y,t)v(y,t)dy,
\end{aligned}
$$

which, combining (2.22), implies that there exists constant $C > 0$ such that

$$\|v_x\| \le C\|w\|$$

and then by (2.23)

$$\|u\|_{H^1} \le \|w\|_{H^1} + \|v\|_{H^1} \le C\|w\|_{H^1}.$$

By analogy, we can show that $K^{-1} : H^2 : XT_0 \to H^2 : XT_0$ is bounded.

To prove *2.*: that the transformation (2.20) converts the system (2.3) and (2.5) and system (2.4) and (2.5) into (2.9) and (2.10), respectively, we compute as follows.

$$
\begin{aligned}
w_t(x,t) &= u_t(x,t) + \int_0^x k(x,y,t)u_t(y,t)dy + \int_0^x k_t(x,y,t)u(y,t)dy \\
&= u_t(x,t) + \int_0^x k(x,y,t)[u_{yy}(y,t) + a(y,t)u(y,t)]dy + \int_0^x k_t(x,y,t)u(y,t)dy \\
&= u_t(x,t) + k(x,x,t)u_x(x,t) - k(x,0,t)u_x(0,t) \\
&\quad - k_y(x,x,t)u(x,t) + k_y(x,0,t)u(0,t) \\
&\quad + \int_0^x [k_{yy}(x,y,t) + k_t(x,y,t) + k(x,y,t)a(y,t)]u(y,t)dy, \tag{2.24}
\end{aligned}
$$

$$w_x(x,t) = u_x(x,t) + k(x,x,t)u(x,t) + \int_0^x k_x(x,y,t)u(y,t)dy, \tag{2.25}$$

$$
\begin{aligned}
w_{xx}(x,t) &= u_{xx}(x,t) + \frac{d}{dx}(k(x,x,t))u(x,t) + k(x,x,t))u_x(x,t) \\
&\quad + k_x(x,x,t)u(x,t) + \int_0^x k_{xx}(x,y,t)u(y,t)dy. \tag{2.26}
\end{aligned}
$$

It then follows from (2.2) and (2.5) that

$$
\begin{aligned}
& w_t - w_{xx} + \lambda w \\
&= u_t(x,t) + k(x,x,t)u_x(x,t) - k(x,0,t)u_x(0,t) \\
&\quad - k_y(x,x,t)u(x,t) + k_y(x,0,t)u(0,t) \\
&\quad + \int_0^x [k_{yy}(x,y,t) + k_t(x,y,t) + k(x,y,t)a(y,t)]u(y,t)dy \\
&\quad - u_{xx}(x,t) - \frac{d}{dx}(k(x,x,t))u(x,t) - k(x,x,t))u_x(x,t) \\
&\quad - k_x(x,x,t)u(x,t) - \int_0^x k_{xx}(x,y,t)u(y,t)dy \\
&\quad + \lambda u(x,t) + \lambda \int_0^x k(x,y,t)u(y,t)dy \\
&= \left(a(x,t) - k_x(x,x,t) - k_y(x,x,t) - \frac{d}{dx}(k(x,x,t)) + \lambda \right) u(x,t) \\
&\quad + k_y(x,0,t)u(0,t) - k(x,0,t)u_x(0,t) \\
&\quad + \int_0^x [k_{yy}(x,y,t) + k_t(x,y,t) - k_{xx}(x,y,t) + (a(y,t) + \lambda)k(x,y,t)]\, u(y,t)dy \\
&= 0. \hspace{8cm} (2.27)
\end{aligned}
$$

By the boundary condition of (2.5), we deduce that $w(0,t) = 0$. Using feedback law (2.3) or (2.4), we obtain

$$
w(1,t) = u(1,t) + \int_0^1 k(1,y,t)u(y,t)dy = 0,
$$

or

$$
w_x(1,t) = u_x(1,t) + k(1,1)u(1,t) + \int_0^1 k_x(1,y,t)u(y,t)dy = 0.
$$

$\square$

Remark 2.2. *We require $t \in (0,t_0)(t_0 < 1)$ in this lemma since k is not bounded when t approaches 1^-. However, this requirement is tolerable since in practice, we only require the system to be stable in a time interval $t \in (0,T_0)$ and this can be rescaled to $(0,t_0)$ by $t \to t/T(T > T_0)$ in the heat equation. Moreover, T_0 can be choose as big as we want.*

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We first note that problem (2.5) with either (2.3) or (2.4) is well posed since, by Lemma 2.2, they can be transformed to the problem (2.9) or (2.10) via the isomorphism defined by (2.20) and the problem (2.9) or (2.10) is well posed (see, e.g., [9, Chap. IV]). Moreover, there exists a positive constant $C > 0$ such that

$$
\begin{aligned}
\|u(t)\|_{H^1} &\le C\|w(t)\|_{H^1}, \\
\|w^0\|_{H^1} &\le C\|u^0\|_{H^1}.
\end{aligned}
$$

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Therefore, it is sufficient to prove (2.8) for the solution w of (2.9) or (2.10). We do so only for problem (2.9) since the situation for problem (2.10) is similar.

We define the energy

$$E(t) = \frac{1}{2} \int_0^1 w(x,t)^2 dx.$$

Multiplying the first equation of (2.9) by w and integrating from 0 to 1 by parts we get

$$
\begin{aligned}
\dot{E}(t) &= w_x w \Big|_0^1 - \int_0^1 w_x(x,t)^2 dx - \lambda \int_0^1 w(x,t)^2 dx \\
&= - \int_0^1 w_x(x,t)^2 dx - \lambda \int_0^1 w(x,t)^2 dx \\
&\leq -2\lambda E(t),
\end{aligned}
$$

which implies

$$E(t) \leq E(0)e^{-2\lambda t}, \quad \text{for } t \geq 0.$$

Set

$$V(t) = \int_0^1 w_x(x,t)^2 dx.$$

Multiplying the first equation of (2.9) by w_{xx} and integrating from 0 to 1 by parts we obtain

$$
\begin{aligned}
\dot{V}(t) &= -2 \int_0^1 w_{xx}^2 dx + 2\lambda \int_0^1 w w_{xx} dx \\
&= -2 \int_0^1 w_{xx}^2 dx - 2\lambda \int_0^1 w_x^2 dx \\
&\leq -2\lambda V(t),
\end{aligned}
$$

which implies that

$$V(t) \leq V(0)e^{-2\lambda t}.$$

This shows that (2.8) holds. $\qquad \square$

3 Neumann Boundary Conditions

To stabilize the Neumann boundary value problem

$$
\begin{cases}
u_t(x,t) = u_{xx}(x,t) + a(x,t)u(x,t) & \text{in } XT, \\
u_x(0,t) = u_x(1,t) = 0 & \text{in } T,
\end{cases}
\tag{3.1}
$$

we consider the problem

$$
\begin{cases}
k_{xx}(x,y,t) - k_{yy}(x,y,t) = (a(y,t)+\lambda)k(x,y,t), & \text{in } \Omega T \\
k_y(x,0) = 0, & \text{in } X \\
k_x(x,x,t) + k_y(x,x,t) + \frac{d}{dx}(k(x,x,t)) = a(x,t) + \lambda, & \text{in } XT
\end{cases}
\tag{3.2}
$$

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where λ is any constant. Using the solution k, we then obtain a Dirichlet boundary feedback law

$$u(1,t) = -\int_0^1 k(1,y,t)u(y,t)dy \qquad \text{in } T \qquad (3.3)$$

and Neumann boundary feedback law

$$u_x(1,t) = -k(1,1)u(1,t) - \int_0^1 k_x(1,y,t)u(y,t)dy \qquad \text{in } T. \qquad (3.4)$$

With one of the boundary feedback laws, the system

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + a(x,t)u(x,t) & \text{in } XT, \\ u_x(0,t) = 0 & \text{in } T, \\ u(x,0) = u^0(x) & \text{in } X \end{cases} \qquad (3.5)$$

is exponentially stable. To state this result, we introduce the compatible conditions for the initial data

$$u_x^0(0) = 0, \quad u^0(1,t) = -\int_0^1 k(1,y,t)u^0(y)dy \qquad (3.6)$$

$$u_x^0(0) = 0, \quad u_x^0(1,t) = -k(1,1)u^0(1,t) - \int_0^1 k_x(1,y,t)u^0(y)dy \qquad (3.7)$$

Theorem 3.1. *Assume that $\lambda > 0$ is any positive constant and $a(x,t)$ is continuous in x and analytic in t uniformly in x. For arbitrary initial data $u^0(x) \in H^1(0,1)$ with the compatible condition (3.6) or (3.7), equation (3.5) with either (3.3) or (3.4) has a unique solution that satisfies*

$$\|u(t)\|_{H^1} \le M\|u^0\|_{H^1}e^{-\lambda t}, \quad t \in (0,t_0)(t_0 < 1)$$

where M is a positive constant independent of u^0.

Proof. The proof is the same as that of Theorem 2.1. Only thing we need to do is to show that problem (3.2) has a unique solution. This is also considered by Colton [7] and his theorem is quoted as Lemma 3.1 below. $\square$

Lemma 3.1. (Colton) *Suppose that $a(x,t)$ is continuous in x and analytic in t uniformly in x. Then problem (3.2) has a unique solution which is twice continuously differentiable in x, y.*

Similar to Lemma 2.2, we have

Lemma 3.2. *Let $k(x,y,t)$ be the solution of problem (3.2) and define the linear bounded operator $K : H^i:XT_0 \to H^i:XT_0 \ (i = 0,1,2)$ by*

$$w(x,t) = (Ku)(x,t) = u(x,t) + \int_0^x k(x,y,t)u(y,t)dy.$$

Then,

1. *K has a linear bounded inverse $K^{-1} : H^i{:}XT_0 \to H^i{:}XT_0$ $(i = 0, 1, 2)$.*

2. *K converts the system (3.3) and (3.5) and system (3.4) and (3.5) into*

$$\begin{cases} w_t = w_{xx} - \lambda w & in\ XT, \\ w_x(0, t) = w(1, t) = 0 & in\ T, \\ w(x, 0) = w^0(x) & in\ X, \end{cases}$$

or

$$\begin{cases} w_t = w_{xx} - \lambda w & in\ XT, \\ w_x(0, t) = w_x(1, t) = 0 & in\ T, \\ w(x, 0) = w^0(x) & in\ X, \end{cases}$$

respectively, where $w^0(x) = u^0(x) + \int_0^x k(x, y, t)u^0(y)dy$.

4 Simulations

To see how the boundary control stabilize the heat equation, we studied the case of Dirichlet boundary problem (2.1) in which

$$a(x, t) = x(bt + c), \tag{4.1}$$

where $b > 0$ and $c > 0$ are positive constants. Both the controlled system by the Dirichlet boundary control (2.3) and uncontrolled systems are studied. In all the simulation, we use the initial data

$$u^0(x) = 10 \left(\frac{1}{4} - \left(x - \frac{1}{2} \right)^2 \right) \sin(4\pi x) + 5 \left(\frac{1}{4} - \left(x - \frac{1}{2} \right)^2 \right), \tag{4.2}$$

which is arbitrarily chosen. If not stated explicitly, the following values are used in the simulation: $\lambda = 10$, $b = 200$, $c = 5$. The x grid is set to be 100 and the time step size is set to be 10^{-5} but when the figures are drawn, we used larger step sizes in x and t for clarity.

The corresponding model of the simulation can be viewed as the heat diffusion along a one dimensional rod, during which heat is also generated unevenly and increasingly. The temperature of the left end of this rod is fixed zero and another is allowed to adjust to let the temperature of the rod converge to zero.

To reveal the various features and relations of the uncontrolled system, controlled system and the control inputs, we do the following simulations. We studied the effects of different b's in $a(x, t)$ with $b = 200$ as shown in figure (5.1.a) and $b = 150$ in figure (5.1.c) to the uncontrolled system (2.1). Clearly, (1) the solution to the uncontrolled system (2.1) is really divergent with time going and (2) the larger the coefficient function $a(x, t)$ is (here the bigger the b is), the faster the solution diverges. The solution to the controlled system with $b = 200$ is shown in Figure (5.1.b). It is seen that the feedback control can really stabilize the system.

We also studied the effect of different λ's in the kernel k, with $\lambda = 10$ shown in figure (5.1.b) and $\lambda = 40$ in figure (5.1.d). Comparing the $x = 1$ edge of these two figures, we can find that the larger the λ is, the faster the solution converges to zero, as expected from equation (2.8). In all the simulations of (5.1.a-d), the kernels are taken to be the first 3 terms in it series approximation (2.18), that is $G_0 + G_1 + G_2$. In (5.1.e), we showed the difference of solutions to the system controlled by a kernel of first three terms ((5.1.b)) and the system controlled by a kernel of first four terms. If denoting these two solutions by $u_3(x, t)$ and $u_4(x, t)$, in figure (5.1.e) we showed $(u_3(x, t) - u_4(x, t))$, from which we can see that the high order approximated kernel can stabilize the system in a faster way. However, this is not a general conclusion for all $a(x, t)$ since in our case, all terms of G_n are positive and thus the more terms you use, the better approximation and better control it gives.

5   Remarks

As a future work, one can try to stabilize the nonlinear problem

$$u_t(x, t) = u_{xx}(x, t) + a(x, t, u(x, t))u(x, t)$$

where a depends not only on x, t but also on u itself. However, we don't know how to prove the unique existence of the corresponding kernel. Once the unique existence of the corresponding kernel was shown, all the results in section (2) and (3) may be generalize to this case.

Another interesting result arises from the simulation. We tried to calculate the solution to the controlled system (by 3 terms kernel) after $t = 1$, which is out of the time domain in our theorems. From figure (5.1.f), one can see the system is still stable. This implies that the condition that $a(x, t)$ should be analytic in t in theorems (2.1) and (3.1) is only sufficient but not necessary.

Acknowledgment

References

[1] D. M. Boskovic, M. Krstic and W. J. Liu, Boundary control of an unstable heat equation via measurement of domain-averaged temperature. *IEEE Trans. Automatic Control*, vol.46, 2001, no. 12, 2022-2028.

[2] A. Balogh and M. Krstic, Infinite-step backstepping for a heat equation-like PDE with arbitrarily many unstable eigenvalues. *Proceedings of the 2001 American Control Conference*, pp. 2480–2485.

[3] A. Balogh and M. Krstic, Infinite dimensional backstepping – style feedback transformations for a heat equation with an arbitrary level of instability. *European Journal of Control*, vol. 8, 2002, pp. 165–176.

[4] Weijiu Liu, Boundary Feedback Stabilization of an Unstable Heat Equation, *SIAM J. Cont. Optim.* vol. 42, pp. 1033-1043.

[5] Andrey Smyshlyaev and Miroslav Krstic, On Control Design for PDEs with Space-Dependent Diffusivity and Time-Dependent Reactivity, accepted by *Automatica*.

[6] R. Adams, *Sobolev Spaces*. Academic Press, New York, 1975.

[7] D. L. Colton, *Analytic Theory of Partial Differential Equations*, 1980, Pitman Press, London, pp: 95-102. Also see: David Colton, The Solution of Initial-Boundary Value Problems for Parabolic Equations by the Method of Integral Operators, *J. Diff. Equ.* vol. 26, 1977, 181-190.

[8] Thomas I. Siedman, Two Results on Exact Boundary Control of Parabolic Equations, *Appl. Math. Optim.* vol. 11, 1984, 145-152.

[9] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'ceva, *Linear and Quasi–linear Equations of Parabolic Type,* Translations of AMS, vol. 23, 1968.

[10] J.L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications, Vol.I,* Springer-Verlag, Berlin, Heidelberg, New York, 1972.
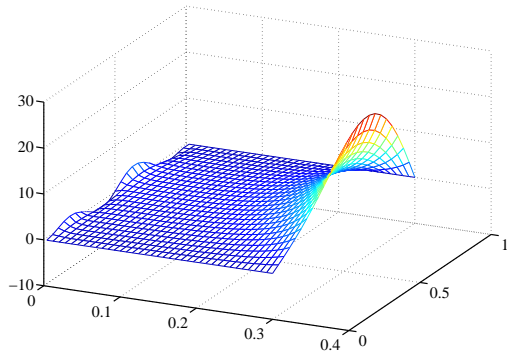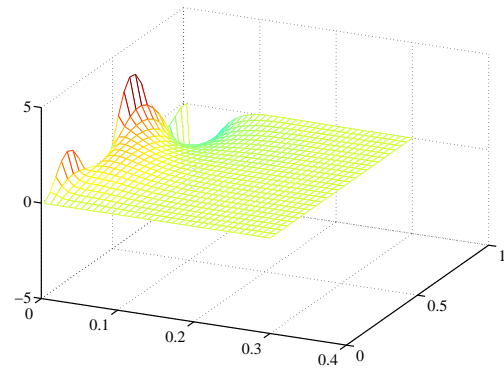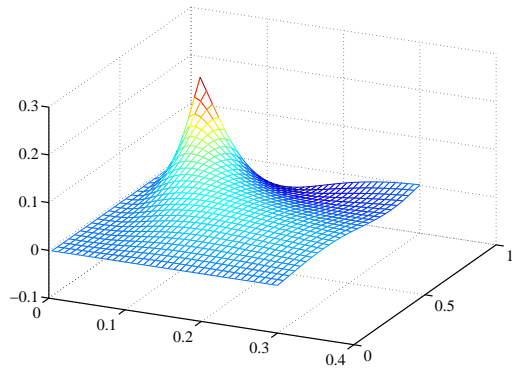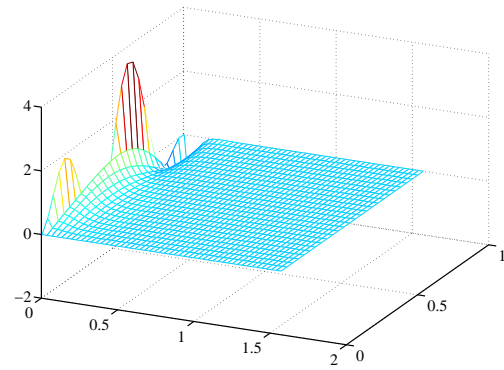
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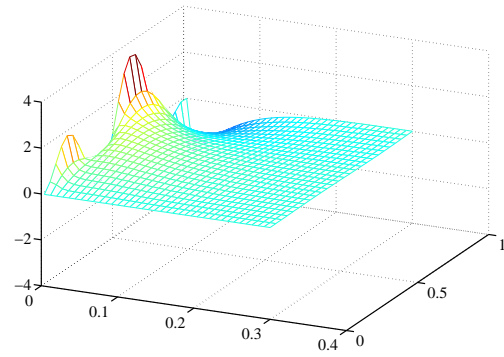
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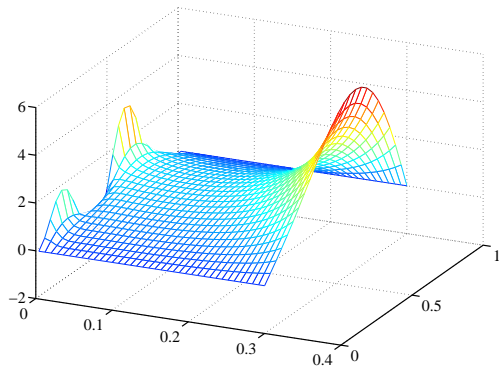
(c)

(d)

(e)

(f)

Figure 5.1: The temperature of the uncontrolled systems, controlled systems vs. time; See text for explanation.